Erasmus University Rotterdam

# On super-level sets of conditional multivariate densities for multiple-output quantile regression *


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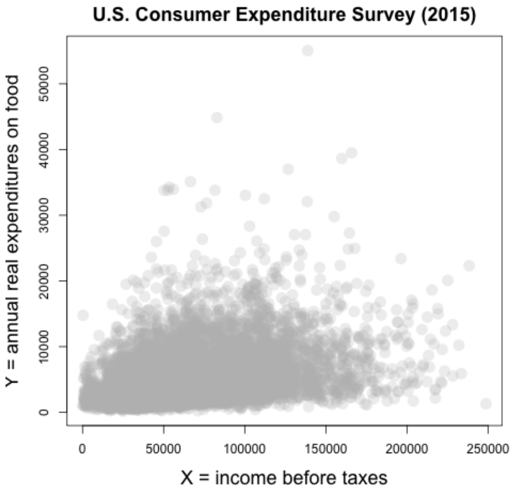
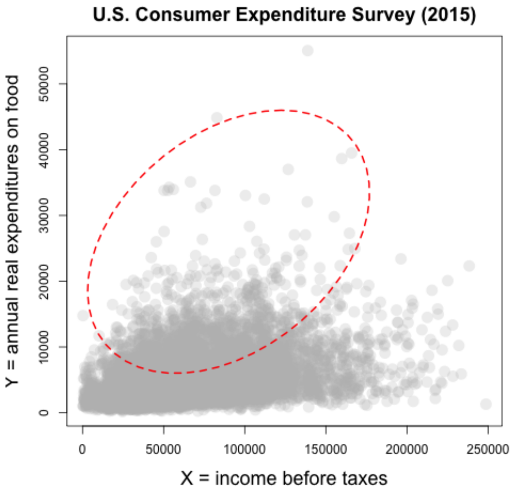
# Example: Heterogeneity in Household Consumption Patterns




- $\{y_n, x_n\}_{n=1}^{N}$ and $N = 29,988$

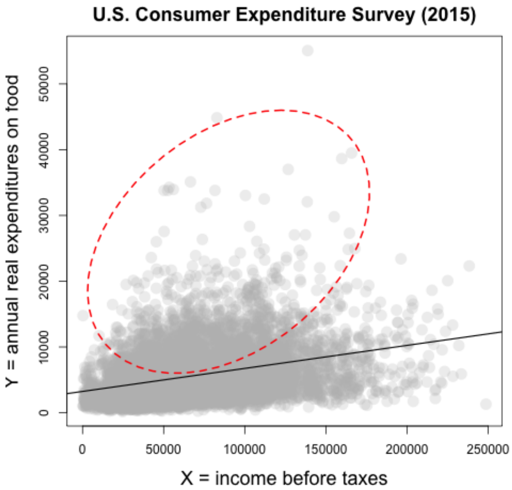
# Example: Heterogeneity in Household Consumption Patterns




- $\{y_n, x_n\}_{n=1}^{N}$ and $N = 29,988$
- $y_n = \mu + x_n'\beta + u_n$

Assume, our interest is in the "top" (i.e., 10%) household segment.

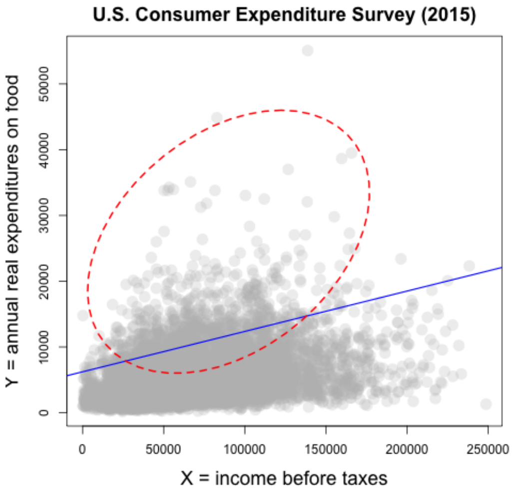
# Example: Heterogeneity in Household Consumption Patterns




Mean regression

$\overline{m}(x_n) = \mu + x_n' \beta$

# Example: Heterogeneity in Household Consumption Patterns

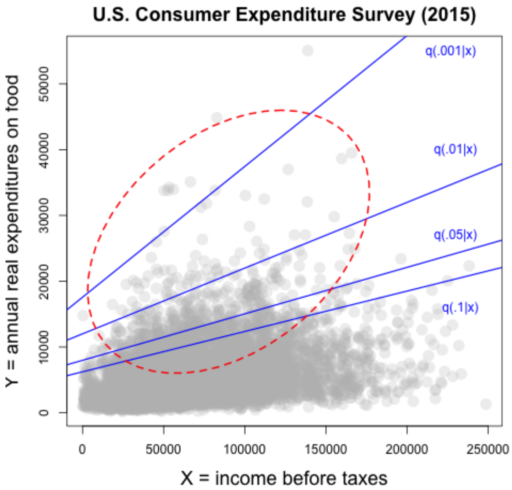



Quantile regression

$$q(.1|x_n) = \mu_{(.1)} + x_n'\beta_{(.1)}$$

Provides insights into the effects of covariates that are missed with mean regression!

# Example: Heterogeneity in Household Consumption Patterns




Quantile regression

$$\left\{ q(\alpha_l | x_n) = \mu_{(\alpha_l)} + x_n' \beta_{(\alpha_l)} \right\}_{l=1}^{L}$$

Provides a comprehensive picture of the conditional response distribution!

# Contributions in Economics

**Study of heterogeneity in treatment participation**
(e.g., Abadie, Angrist & Imbens, 2002; Athey & Imbens 2006; Firpo, 2007; Chernozhukov & Hansen, 2013)

**Value-at-risk (tail value) measurement**
(Chernozukov & Umantsev, 2001; Engle & Manganelli, 2004)

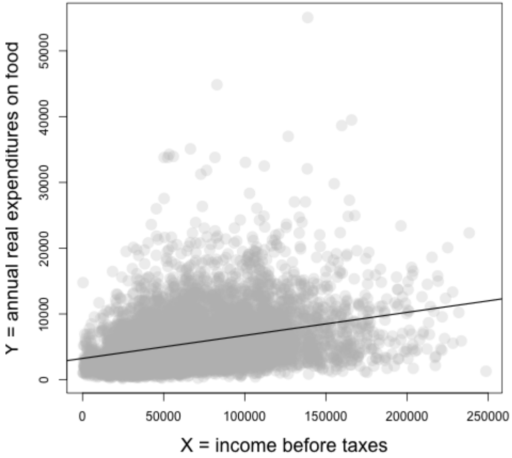
**Exploration of multiple-output and functional responses**
(Hallin, Paindaveine & Siman, 2010; Wei 2008; Carlier, Chernozhukov & Galichon, 2016, Hallin & Siman, 2016; **this talk!**)

# This Talk

- The quantile is defined as a property of an (estimated) conditional multivariate density.
- This so-called super-level set enables a clear probabilistic interpretation and enjoys favorable quantile properties.
- Linear and non-linear multivariate as well as univariate regression quantiles are obtained in a comprehensive (fully) Bayesian framework.

# Regression Quantiles in a Nutshell

(Conditional) mean regression

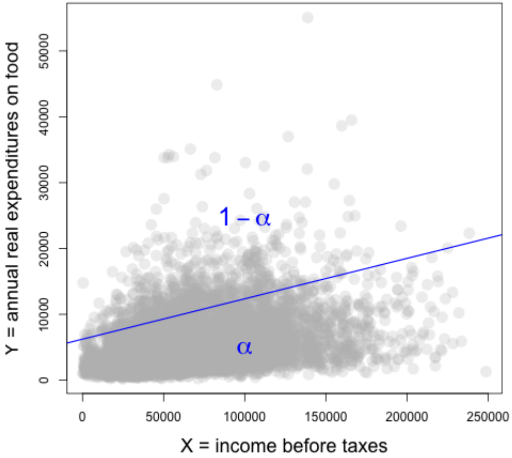



$$\underset{\mu,\beta}{\text{argmin}} \sum_{n=1}^{N} |y_n - \overline{m}(x_n)|^2$$

$\Rightarrow$ solved with numerical linear algebra (i.e., OLS).

# Regression Quantiles in a Nutshell

(Conditional) quantile regression



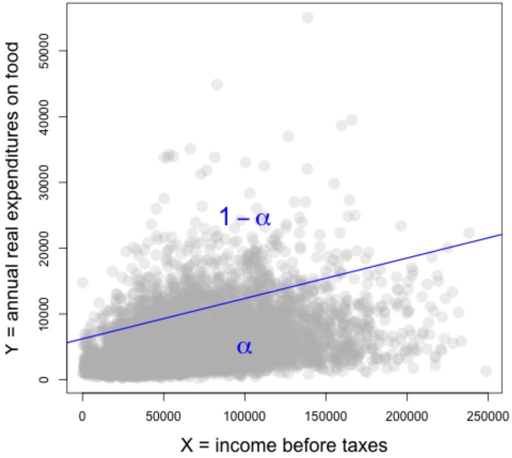

$$\underset{\mu_{(\alpha)},\beta_{(\alpha)}}{\text{argmin}} \sum_{n=1}^{N} \rho_\alpha |y_n - q(\alpha|x_n)|$$

for $0 < \alpha < 1$, and asymmetric weight ("check") function:

$$\rho_\alpha(u_n) = \begin{cases} \alpha u_n, & u_n > 0 \\ -(1-\alpha)u_n, & u_n \leq 0 \end{cases}$$

$\Rightarrow$ solved with linear programming (see, Koenker & Bassett, 1978).

(Conditional) quantile regression




$$\rho_\alpha(u_n) = u_n(\alpha I_{(u_n>0)} - (1-\alpha) I_{(u_n \leq 0)})$$

equivalent to:

$$u_n \overset{iid}{\sim} \mathcal{AL}(0, \sigma, \alpha)$$

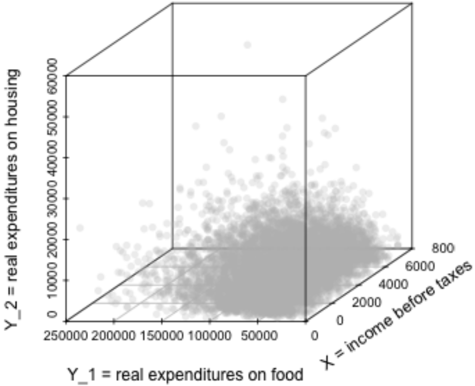
with asymmetry parameter $0 < \alpha < 1$

Example

$\Rightarrow$ ML (and Bayesian) inference is straightforward (see, Koenker & Machado, 1999; Yu & Moyeed, 2001).

# Regression Quantiles in a Nutshell

Multiple-outputs: $\boldsymbol{y}_n = (y_{n1}, \ldots, y_{nK})'$




$$\boldsymbol{q}(\alpha|\boldsymbol{x}_n) = \boldsymbol{\mu}_{(\alpha)} + \boldsymbol{B}_{(\alpha)}\boldsymbol{x}_n$$

where $\boldsymbol{\mu}_{(\alpha)}$ is $K \times 1$ and $\boldsymbol{B}_{(\alpha)}$ is $K \times G$

$\Rightarrow$ Seemingly unrelated regression, simultaneous equations, VAR ...

The conditional Koenker-Bassett quantile concept is not easy to extend!

# Multivariate Quantiles

**Attempt 1: Conditioning**

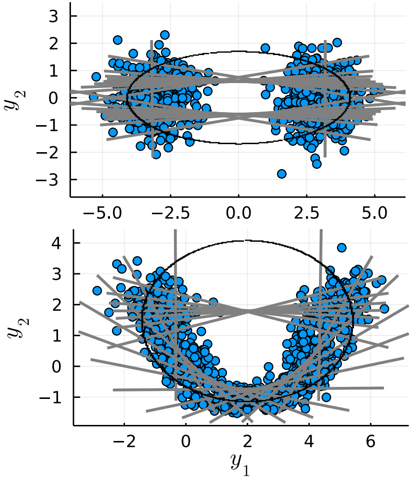
$$\boldsymbol{q}(\alpha) = [q_{y_{n1}}(\alpha|\boldsymbol{y}_{n(-1)}), \ldots, q_{y_{nK}}(\alpha|\boldsymbol{y}_{n(-K)})]'$$

- Input-space augmentation, assumes all "regressors" are fixed!

## Attempt 2: Directional

- Convex intersection of $\alpha$-quantile halfspaces for different (Koenker-Bassett) regression hyperplanes (Hallin, Paindaveine & Siman, 2010).

Areas within the gray lines (contours) give the 80%-**directional quantile** (20 directions).



# Multivariate Quantiles

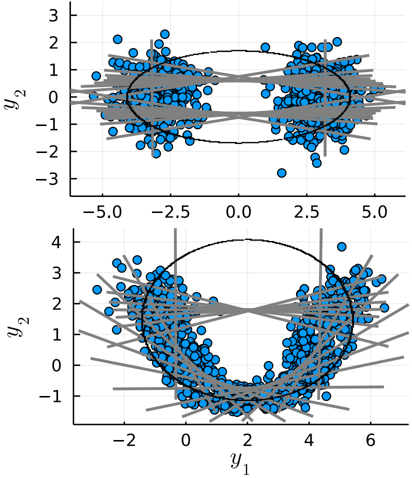
**Attempt 2: Directional**

- Convex intersection of $\alpha$-quantile halfspaces for different (Koenker-Bassett) regression hyperplanes (Hallin, Paindaveine & Siman, 2010).

**Attempt 3: Direct**

- Find an ellipsoid around a (determined) center with $\alpha$-probability mass (e.g., Hallin & Siman, 2016).

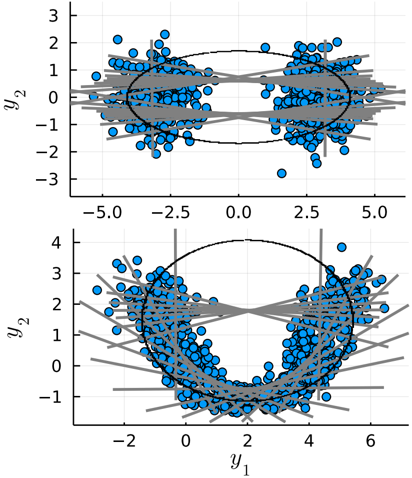
Areas within the black lines (countours) give the 80%-**elliptical quantile**.



# Multivariate Quantiles

- The (directional) quantile contours are not guaranteed to cover $\alpha$.
- The quantile regions can cover large parts with little to no probability mass.
- The definitions cannot easily be extended (i.e., to more than two outputs/ inputs).

Interpretability and practical use?



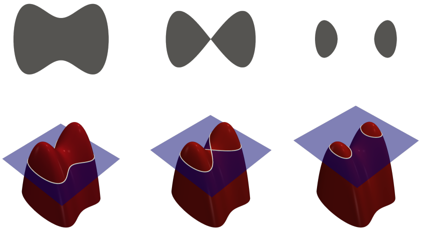
# Multivariate Quantiles

**Level set**

$$\mathcal{L}(f;t) = \left\{ \boldsymbol{y}_n \in \mathbb{R}^K : f(\boldsymbol{y}_n) = t \right\}$$

$\Rightarrow$ Cross-section of $f(\cdot)$ at a given (constant) value $t$ (Osher & Sethian, 1988).

Level sets of a bivariate bimodal distribution for three different values of $t$.



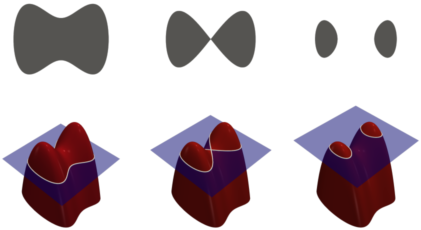
# Multivariate Quantiles

**Super-level set**

$$\mathcal{L}(f;t) = \left\{ \boldsymbol{y}_n \in \mathbb{R}^K : f(\boldsymbol{y}_n) \geq t \right\}$$

for threshold $t > 0$, gives the highest density region for $f(\cdot)$ (see, e.g., Hartigan, 1987).

Level sets of a bivariate bimodal distribution for three different values of $t$.



# Multivariate Quantiles

**Super-level set**

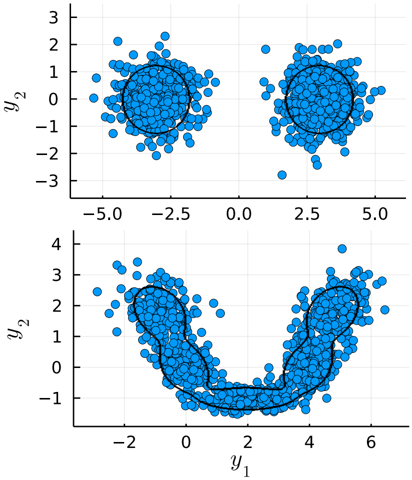
$$\mathcal{L}(f;t) = \left\{ \boldsymbol{y}_n \in \mathbb{R}^K : f(\boldsymbol{y}_n) \geq t \right\}$$

for threshold $t > 0$, gives the highest density region for $f(\cdot)$ (see, e.g., Hartigan, 1987).

**Super-level set quantile**

$$\boldsymbol{q}(\alpha) = \mathcal{L}(f;t^\star_\alpha),$$
$$t^\star_\alpha = \text{sup}\left\{ Pr(\boldsymbol{y}_n \in \mathcal{L}(f;t)) \geq \alpha \right\}$$

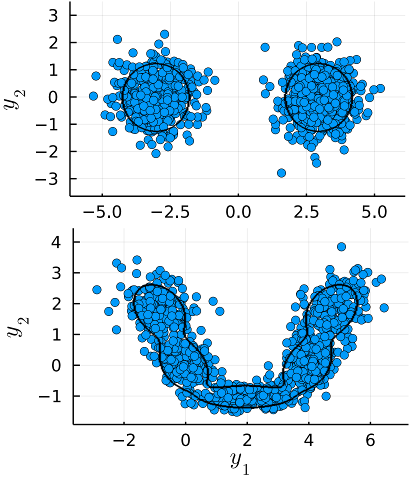
Areas within the lines (quantile contours) correspond to 80% probability mass.



# Multivariate Quantiles

- Supports a clear probabilistic interpretation (in terms of $\alpha$).
- (Flexible) quantile regions cover areas with high probability mass.
- Extensions to more than two inputs/ outputs are straightforward.

Areas within the lines (quantile contours) correspond to 80% probability mass.



# Properties

| | Univariate quantile collection | Directional quantile | Elliptical quantile | Reference quantile (Wei, 2008) | Super-level set quantile |
|---|---|---|---|---|---|
| Vector-valued | no | ✓ | ✓ | ✓ | ✓ |
| $\alpha$-probability coverage control | no | no | ✓ | ✓ | ✓ |
| Reflects probability mass concentration | no | no | no | no | ✓ |
| Nestedness | no | ✓ | ✓ | ✓ | ✓ |
| Uniqueness | ✓ | ✓ | ✓ | ✓ | ✓ |
| Affine equivariance | no | ✓ | ✓ | ✓ | ✓ |
| Invariant to coordinate transformations | ✓ | no | no | no | no |
| Invariant to monotone transformations | no | no | no | no | no |
| Reduces to classic univariate quantile | ✓ | no | no | no | no |

# HPD vs. Super-Level Set Quantiles

**Highest posterior density set**

- Operationalizes uncertainty of a model parameter for a (typically) univariate and unimodal posterior distribution (i.e., an interval, Box & Tiao, 1965).

**Super-level set**

- Quantifies uncertainty in a (set of) response variable(s), conditional on other response variables, for arbitrarily shaped joint posterior distributions (i.e., an interval or a set of intervals).

# Let's go (fully) Bayesian

(Overfitted) Finite Gaussian Mixture Model:

$$f\left(\boldsymbol{y}_n|\boldsymbol{x}_n\right) = \sum_{m=1}^{M} \kappa_m \phi\left(\boldsymbol{g}_m(\boldsymbol{x}_n), \boldsymbol{\Sigma}_m\right),$$

$$\text{where } \boldsymbol{g}_m(\boldsymbol{x}_n) = \boldsymbol{\mu}_m + \boldsymbol{B}_m \boldsymbol{x}_n,$$

$$\boldsymbol{\kappa}|\{\bar{\rho}_m\} \sim \mathcal{D}\left(\bar{\rho}_1, \cdots, \bar{\rho}_M\right),$$

$$\bar{\rho}_m \sim \mathcal{G}\left(\underline{a}_1, 1/(\underline{a}_2 M)\right).$$

with $M$ comparatively large (Nobile & Fearnside, 2007; Rousseau & Mengersen, 2011) and a Shrinkage Prior on $\phi(\boldsymbol{g}_m(\boldsymbol{x}_n), \boldsymbol{\Sigma}_m)$ (Malsiner-Walli, Frühwirth-Schnatter & Grün, 2016).

# Implementation

$$\boldsymbol{\mu}_m^{\mathcal{K}|\mathcal{C}}(\boldsymbol{y}_{\mathcal{C}}, \boldsymbol{x}) = \boldsymbol{g}_{m,\mathcal{K}}(\boldsymbol{x}) + \boldsymbol{\Sigma}_{m,\mathcal{K},\mathcal{C}} \boldsymbol{\Sigma}_{m,\mathcal{C},\mathcal{C}}^{-1} \Big( \boldsymbol{y}_{\mathcal{C}} - \boldsymbol{g}_{m,\mathcal{C}}(\boldsymbol{x}) \Big),$$

$$\boldsymbol{\Sigma}_m^{\mathcal{K}|\mathcal{C}} = \boldsymbol{\Sigma}_{m,\mathcal{K},\mathcal{K}} - \boldsymbol{\Sigma}_{m,\mathcal{K},\mathcal{C}} \boldsymbol{\Sigma}_{m,\mathcal{C},\mathcal{C}}^{-1} \boldsymbol{\Sigma}_{m,\mathcal{C},\mathcal{K}} \ ,$$

$$\omega_m^{\mathcal{C}}(\boldsymbol{y}_{\mathcal{C}}, \boldsymbol{x}) = \frac{\kappa_m \phi\Big(\boldsymbol{g}_{m,\mathcal{C}}(\boldsymbol{x}), \boldsymbol{\Sigma}_{m,\mathcal{C},\mathcal{C}}\Big)}{\sum_{l=1}^{M} \kappa_l \phi\Big(\boldsymbol{g}_{l,\mathcal{C}}(\boldsymbol{x}), \boldsymbol{\Sigma}_{l,\mathcal{C},\mathcal{C}}\Big)} \ ,$$

... serve as inputs to the Level-set algorithm to compute $\widetilde{\boldsymbol{q}}(\alpha)$.

# Simulation Exercise

1. Multivariate Gaussian
2. Multivariate Student-t
3. Multivariate log-Gaussian
4. Conditional heteroskedasticity
5. Multivariate Gaussian mixture

$1,000$ data sets with a sample size of $10,000$ for each DGP.

Hyperparameters

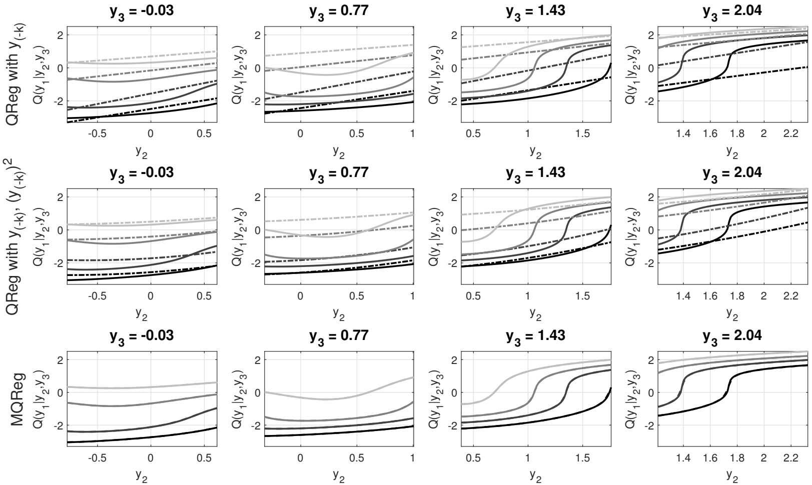
- $M = 5$
- $\underline{a}_1 = 10$, $\underline{a}_2 = 40$ (Dirichlet prior)
- $\underline{b}_1 = .5$, $\underline{b}_2 = .5$ (Gamma prior)

MCMC samples

- Effective: 50,000 / 10 (Burn-in: 10,000)

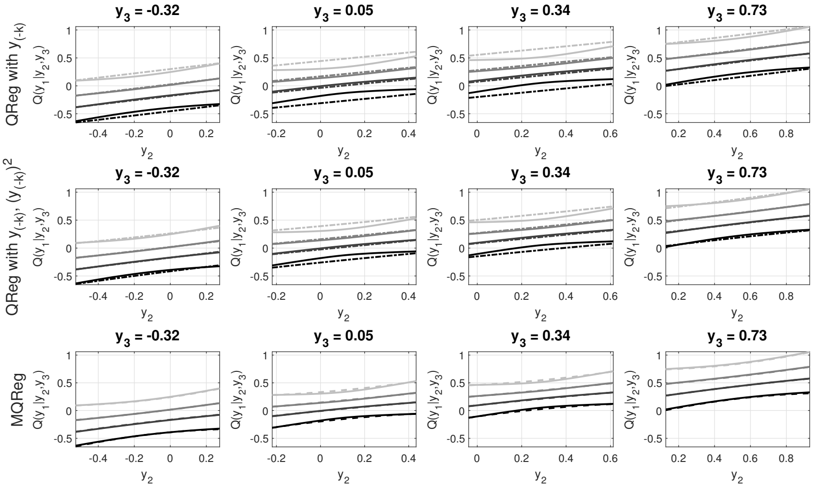
# Simulation Exercise: Multivariate Gaussian Mixture

Estimated (dashed lines) vs. true (solid lines) conditonal quantiles for $\alpha \in \{.2, .4, .6, .8\}$

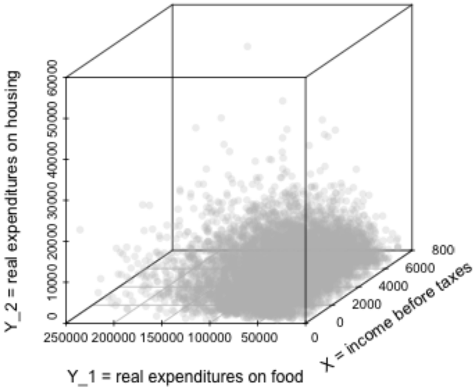


# Simulation Exercise: Conditional Heteroskedasticty

Estimated (dashed lines) vs. true (solid lines) conditonal quantiles for $\alpha \in \{.2, .4, .6, .8\}$



# Heterogeneity in Household Consumption Patterns (cont.)




Hyperparameters

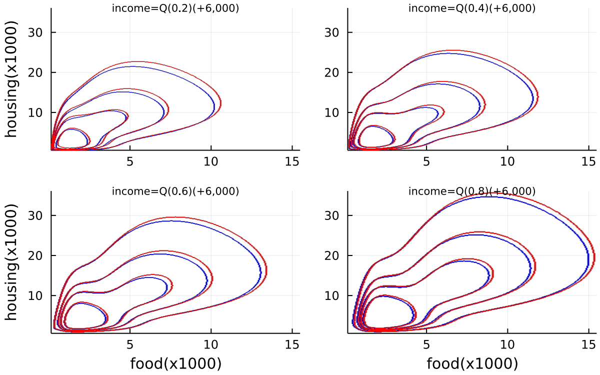
- $M = 5$
- $\underline{a}_1 = 10$, $\underline{a}_2 = 40$ (Dirichlet prior)
- $\underline{b}_1 = .5$, $\underline{b}_2 = .5$ (Gamma prior)

MCMC samples

- Effective: 200,000 / 40 (Burn-in: 400,000)

# Heterogeneity in Household Consumption Patterns (cont.)

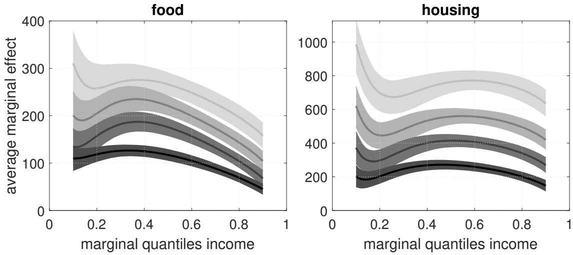
Bivariate quantiles for food and housing conditional on four different income levels. Blue lines corr. to $\alpha \in \{.2, .4, .6, .8\}$. Red lines corr. to an income increase of $6,000. QTE




- $.2$ expenditure quantile does not react considerably.
- $.8$ expenditure ($.2$ income) quantile substantially increases spending.
- No clear substitution patterns between food and housing.

# Heterogeneity in Household Consumption Patterns (cont.)

Quantile-varying marginal effects conditional on income.
Shaded areas give the 90% C.I. for four income $\alpha$-levels.




- Low-income quantile households dedicate most of the additional income to food and shelter.
- Highest-income quantile households hardly increase spendings at all.

# Conclusion

Super-level sets provide a coherent framework for multivariate and univariate conditional as well as marginal quantiles:

- (1) no quantile crossing, (2) flexible quantile contours with exact probability coverage, (3) easy to extend quantile concept.

The overfitted GMM allows for straightforward incorporation of prior information regarding shapes and centers:

- enables a data driven bandwith parameter selection without unpleasant computational features (slow convergence, long runtimes; Polonik, 1997).
- makes no particular residual distribution assumption (see, e.g., Sriram, Ramamoorthi & Ghosh (2013) on the invalidity of the $\mathcal{AL}$-likelihood).

Working paper available via
https://ideas.repec.org/p/tin/wpaper/20220094.html

Many thanks for your attention!

# References

Abadie A, Angrist J, Imbens G (2002). Instrumental variables estimates of subsidized training on the quantile of trainee earnings. *Econometrica*, 70:91–117.

Athey S, Imbens G (2006). Identification and inference in nonlinear difference- in-differences models. *Econometrica*, 74: 431–497.

Brown P J, Griffin J E (2010). Inference with Normal-Gamma prior distributions in regression problems. *Bayesian Analysis*, 5: 171–188.

Carlier G, Chernozhukov V, Galichon A (2016). Vector quantile regression: an optimal transport approach. *The Annals of Statistics*, 44: 1165–1192.

Chernozhukov V, Hansen C (2013). Quantile models with endogeneity. *Annual Review of Economics*, 5:57–81

Chernozhukov V, Umantsev L (2001). Conditional value-at-risk: Aspects of modeling and estimation. *Empirical Economics*, 26: 271–293.

Doksum K (1974). Empirical probability plots and statistical inference for nonlinear models in the two-sample case. *The Annals of Statistics*, 2: 267–277.

Engle R F, Manganelli S (2004). Cariar: Conditional autoregressive value at risk by regression quantiles. *Journal of Business and Economic Statistics*, 22: 367–381.

Firpo S (2007). Efficient semiparametric estimation of quantile treatment effects. *Econometrica*, 75:259–76

Halli M, Paindaveine D, Siman M (2010). Multivariate quantiles and muliple- output regression quantiles: from L1 optimization to halfspace depth. *The Annals of Statistics*, 38: 635–703.

# References


Hallin M, Siman M (2016). Elliptical multiple-output quantile regression and convex optimization. *Statistics & Probability Letters*, 109: 232–237.

Hartigan J A (1987). Estimation of a convex density contour in two dimensions. *Journal of the American Statistical Association*, 82: 267–270.

Koenker R, Bassett G (1978). Regression Quantiles. *Econometrica*, 46: 33–50.

Koenker R, Machado J (1999). Goodness of fit and related inference processes for quantile regression, *Journal of the American Statistical Association*, 94: 1296-309.

Malsiner-Walli G, Frühwirth-Schnatter S, Grün B (2016). Model-based clustering based on sparse finite Gaussian mixtures. *Statistics and Computing*, 26: 303–324.

Nobile A, Fearnside A T (2007). Bayesian finite mixtures with an unknown number of components: The allocation sampler. *Statistics and Computing*, 17: 147–162.

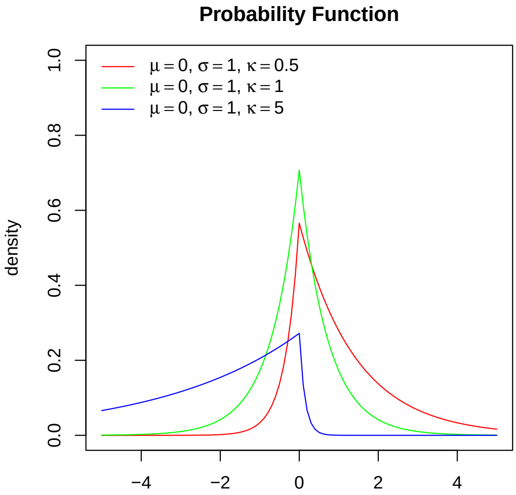
Rousseau J, Mengersen K (2011). Asymptotic behaviour of the posterior distribution in overfitted mixture models. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 73: 689–710.

Taddy M A, Kottas A (2010). A Bayesian nonparametric approach to inference for quantile regression. *Journal of Business & Economic Statistics*, 28: 357–369.

Wei Y (2008). An approach to multivariate covariate-dependent quantile contours with application to bivariate conditional growth charts. *Journal of the American Statistical Association*, 103:, 397–409.

Yu K, Moyeed R A (2001). Bayesian quantile regression. *Statistics & Probability Letters*, 54: 437-447.

# Asymmetric Laplace Distribution



# Prior Distributions

$$\boldsymbol{\mu}_m | \bar{\boldsymbol{v}}_0, \overline{\boldsymbol{V}}_0 \sim \mathcal{N}\left(\bar{\boldsymbol{v}}_0, \overline{\boldsymbol{V}}_0\right),$$

$$\bar{\boldsymbol{v}}_0 \sim \mathcal{N}\left(\underline{\boldsymbol{v}}, \underline{\boldsymbol{V}}\right),$$

$$\underline{\boldsymbol{v}} = median(\boldsymbol{y}_n), \underline{\boldsymbol{V}}^{-1} = \mathbf{0}$$

$$\overline{\boldsymbol{V}}_0 = \text{diag}(R_1^2\lambda_1, \ldots, R_K^2\lambda_K),$$

$$\lambda_k \sim \mathcal{G}\left(\underline{b}_1, 1/\underline{b}_2\right).$$

with response variable-specific value ranges $\{R_k\}$ and local shrinkage factors $\{\lambda_k\}$ $(\underline{b}_1, \underline{b}_2 > 0)$. (see, Brown & Griffin, 2010)

$$\text{vec}(\boldsymbol{B}_m) \sim \mathcal{N}\left(\boldsymbol{c}_0, \boldsymbol{C}_0\right)$$

$$\boldsymbol{\Sigma}_m \sim \mathcal{IW}\left(\boldsymbol{S}_0, s_0\right)$$

where $\boldsymbol{S}_0 = \boldsymbol{I}$ and $s_0 > 2 + K$.

# Sampling Algorithm

- Simulate mixture parameters conditional on $\boldsymbol{z}_n$ $(n = 1, \ldots, N,\ m = 1, \ldots, M)$:
  - Sample $\{\kappa_m\}$ from $\mathcal{D}(\bar{\rho}_1, \ldots, \bar{\rho}_M)$ where $\bar{\rho}_m = \rho_m + N_m$, $N_m = \#\{n : z_n = m\}$.
  - Sample $\{\boldsymbol{\mu}_m\}$ from $\mathcal{N}(\bar{\boldsymbol{v}}_m, \overline{\boldsymbol{V}}_m)$.
  - Sample $\{\boldsymbol{B}_m\}$ from $\mathcal{N}(\boldsymbol{c}_m, \boldsymbol{C}_m)$.
  - Sample $\{\boldsymbol{\Sigma}_m\}$ from $\mathcal{IW}(\boldsymbol{S}_m, s_m)$.
- Sample $z_n$ to classify observations conditional on mixture parameters $(n = 1, \ldots, N)$:
  - $\pi_m \equiv \Pr[z_n = m | \boldsymbol{y}_m, \boldsymbol{\kappa}, \boldsymbol{\mu}, \boldsymbol{B}, \boldsymbol{\Sigma}] \propto \kappa_m \phi\big(\boldsymbol{y}_n; g_m(\boldsymbol{x}_n), \boldsymbol{\Sigma}_m\big)$.
  - Sample $\{z_n\}$ from $\mathcal{M}(\pi_1, \ldots, \pi_M)$.
- Sample hyperparameters:
  - Sample $\{\bar{\rho}_m\}$ simulatneously via a random walk MH-step with proposal density $\log(\rho_m) \sim \mathcal{N}(\log(\rho_m), s^2_{\rho_m})$ from $p(\bar{\rho}_m | \boldsymbol{\kappa}) \propto p(\boldsymbol{\kappa} | \bar{\rho}_m) p(\bar{\rho}_m)$
  - Sample $\{\lambda_k\}$ from $\mathcal{GIG}(\underline{b}_1 - M/2, 2\underline{b}_2, \delta_k)$ where $\delta_k = \sum_{m=1}^{M} (\mu_{m,k} - \bar{v}_{0,k})^2 / R_k^2$.
  - Sample $\bar{\boldsymbol{v}}_0$ from $\mathcal{N}\big(\sum_{m=1}^{M} \boldsymbol{\mu}_k / M, \overline{\boldsymbol{V}}_0 / M\big)$ with $\overline{\boldsymbol{V}}_0 = \mathrm{diag}(R_1^2 \lambda_1, \ldots, R_K^2 \lambda_K)$.

# Super-level Set Algorithm

**Input :** chosen coverage probability $\alpha$
conditional distribution function $F_{\boldsymbol{Y}_\mathcal{K}|\boldsymbol{Y}_\mathcal{C}=\boldsymbol{y}_\mathcal{C}}(\boldsymbol{y})$
grid boundary probability $\epsilon$
dimension-specific grid point number $n_{\text{grid}}$

**Output:** actual coverage probability $p$
numerical quantile $\tilde{Q} = \tilde{Q}_{\boldsymbol{Y}_\mathcal{K}|\boldsymbol{Y}_\mathcal{C}=\boldsymbol{y}_\mathcal{C}}(\alpha)$ of size $n_{\text{grid}}^{|\mathcal{K}|}$

**for** $k \in \mathcal{K}$ **do**
    $\text{grid}_k$ = equally spaced $n_{\text{grid}}$ vector with values from $F^{-1}_{Y_k|\boldsymbol{Y}_\mathcal{C}=\boldsymbol{y}_\mathcal{C}}(\epsilon)$ to $F^{-1}_{Y_k|\boldsymbol{Y}_\mathcal{C}=\boldsymbol{y}_\mathcal{C}}(1-\epsilon)$;
    $\tilde{Q}_{\boldsymbol{Y}_\mathcal{K}|\boldsymbol{Y}_\mathcal{C}=\boldsymbol{y}_\mathcal{C}}(\alpha) = |\mathcal{K}|$-dimensional array of zeros
$P$ = empty $|\mathcal{K}|$-dimensional array to hold probabilities per hypercube
**for** $(i_1 \in 2 : n_{grid}), (i_2 \in 2 : n_{grid}), \ldots, (i_{|\mathcal{K}|} \in 2 : n_{grid})$ **do**
    $P_{i_1, i_2, \ldots, i_{|\mathcal{K}|}} = \Pr[Y_k \in [\text{grid}_{k,i_k-1}, \text{grid}_{k,i_k}]\ \forall k \in \mathcal{K} | \boldsymbol{Y}_\mathcal{C} = \boldsymbol{y}_\mathcal{C}]$
$p = 0$
**while** $p < \alpha$ **do**
    $\mathcal{I}$ = set of indices for which $P$ equals $\max\{P\}$
    $p = p + \sum_{i \in \mathcal{I}} P_i$
    **for** $i \in \mathcal{I}$ **do**
        $\tilde{Q}_i = \alpha$
        $P_i = 0$

# Quantile-Specific Measures

The local marginal effect in the $\alpha$-level quantile of $Y_k$ given $\boldsymbol{Y}_{\mathcal{C}} = \boldsymbol{y}_{\mathcal{C}}$ for a change from $\boldsymbol{x}$ to $\boldsymbol{x} + \Delta_g$ is:

$$\beta^g_{k|\mathcal{C}}(\alpha|\boldsymbol{y}_{\mathcal{C}}, \boldsymbol{x}) = Q_{Y_k|\boldsymbol{Y}_{\mathcal{C}}=\boldsymbol{y}_{\mathcal{C}}}\left(\alpha|\boldsymbol{x} + \boldsymbol{\Delta}_g\right) - Q_{Y_k|\boldsymbol{Y}_{\mathcal{C}}=\boldsymbol{y}_{\mathcal{C}}}\left(\alpha|\boldsymbol{x}\right),$$

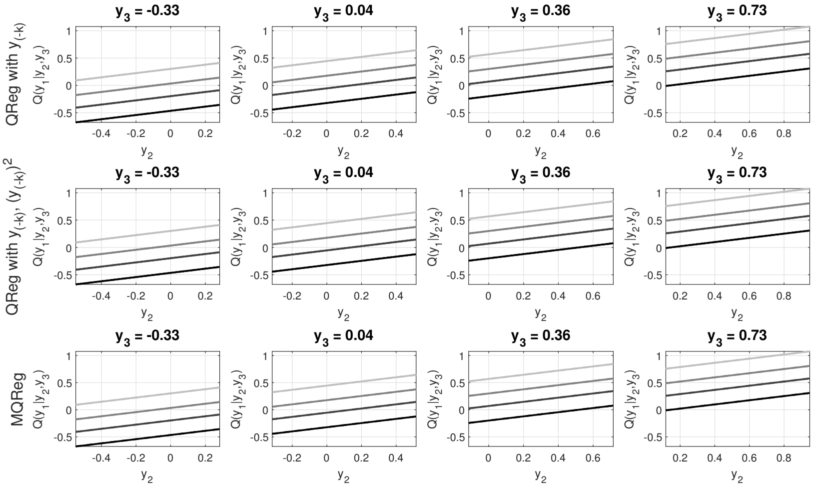
where $\boldsymbol{\Delta}_g$ is a vector with a small value $\delta_g$ at position $g$ and zeros elsewhere (see Doksum, 1974). Back

# Data Generating Processes

1. Multivariate Gaussian:
   $\boldsymbol{y}_n \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} = [.2, .2, .2]'$ and $\Sigma_{jj} = .4$, $\Sigma_{jk} = .25$ $(\forall j \neq k)$.
2. Multivariate Student-t:
   $\boldsymbol{y}_n \sim t_r(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $r = 5$, $\boldsymbol{\mu} = [.2, .2, .2]'$ and $\Sigma_{jj} = .4$, $\Sigma_{jk} = .25$ $(\forall j \neq k)$.
3. Multivariate log-Gaussian:
   $\boldsymbol{y}_n \sim \log \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu} = [.2, .2, .2]'$ and $\Sigma_{jj} = .4$, $\Sigma_{jk} = .25$ $(\forall\ j \neq k)$.
4. Conditional heteroskedasticity:
   $\boldsymbol{y}_n \sim \mathcal{N}(\boldsymbol{\mu}, \Omega_n)$ with $\boldsymbol{\mu} = [.2, .2, .2]'$ and $\Omega_n = \exp(z_n)\boldsymbol{\Sigma}$ where $z_n \sim \mathcal{N}(0, 1)$ and $\Sigma_{jj} = .4$, $\Sigma_{jk} = .25$, $\forall j \neq k$.
5. Multivariate Gaussian mixture:
   $\boldsymbol{y}_n \sim \mathcal{N}(\boldsymbol{\mu}_{z_n}, \boldsymbol{\Sigma}_{z_n})$ with $\Pr[z_n = m] = .33$, for $m = 1, 2, 3$, $\boldsymbol{\mu}_1 = [2, 2, 2]$, $\boldsymbol{\mu}_2 = [0, 0, 0]$, $\boldsymbol{\mu}_3 = [-2, .5, 1]$ and $\Sigma_{1,jj} = .4$, $\Sigma_{1,jk} = .25$, $\boldsymbol{\Sigma}_2 = \boldsymbol{I}$, $\Sigma_{3,jj} = .7$, $\Sigma_{3,jk} = .5$.

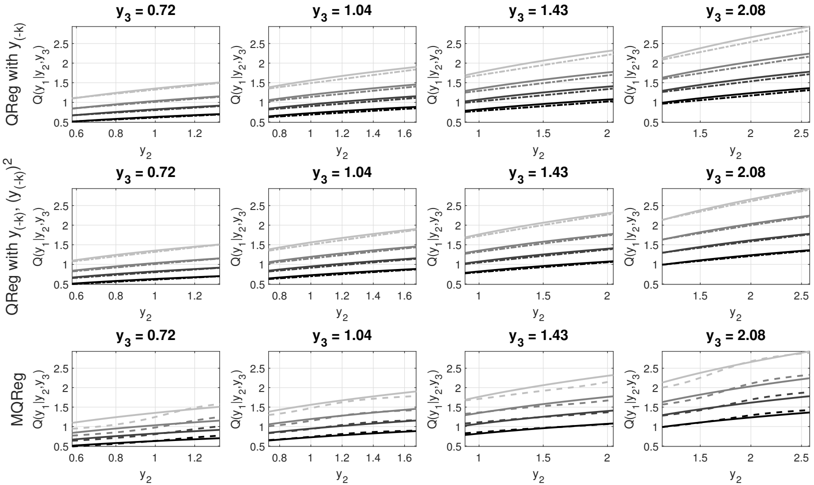
# Simulation Exercise: Multivariate Gaussian

Estimated (dashed lines) vs. true (solid lines) conditonal quantiles for $\alpha \in \{.2, .4, .6, .8\}$



# Simulation Exercise: Multivariate Student-t

Estimated (dashed lines) vs. true (solid lines) conditonal quantiles for $\alpha \in \{.2, .4, .6, .8\}$



# Simulation Exercise: Multivariatae log-Gaussian

Estimated (dashed lines) vs. true (solid lines) conditonal quantiles for $\alpha \in \{.2, .4, .6, .8\}$